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A NOTION OF RANK FOR UNITARY REPRESENTATIONS OF GENERAL LINEAR GROUPS

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ABSTRACT. A notion of rank for unitary representations of general linear groups over a locally compact, nondiscrete field is defined. Rank measures how singular a representation is, when restricted to the unipotent radical of a maximal parabolic subgroup. Irreducible representations of small rank are classified. It is shown how rank determines to a large extent the asymptotic behavior of matrix coefficients of the representations.

I. DEFINITION AND BASIC PROPERTIES OF RANK

0. INTRODUCTION

The unitary dual of $\mathrm{GL}_n(F)$, F a locally compact nondiscrete field, has been classified by D. Vogan [V] in the Archimedean case, and, up to the construction of all cuspidal representations, by M. Tadić in the non-Archimedean case.

Much remains to be done, however, in terms of the study of specific properties of the unitary representations involved. The inspiration for this paper comes form the following well-known theorem of Howe and Moore.

Theorem 0.1 [HM]. *Let G be a Zariski connected reductive algebraic group over a nondiscrete locally compact field F, and let ρ be a unitary representation of dimension greater than 1 such that the center Z of G acts by a character. Then the absolute values of the matrix coefficients of ρ approach zero at infinity on G/Z.*

An easy consequence of this is

Corollary 0.2. *In the hypotheses for Theorem 0.1, if N is a subgroup of G which is not compact modulo Z, any vector that is invariant under N transforms according to a character of G.* □

In particular if ρ is irreducible and has a nontrivial N-invariant vector, ρ must be a character.

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Hence it is possible to deduce a fact about a unitary representation of G (whether it is one-dimensional or infinite-dimensional) just by looking at its N-spectrum.

The notion of rank of a unitary representation, first introduced by R. Howe [H1] for symplectic groups, attempts to refine this idea.

Roughly said, the rank of a unitary representation of GL_n measures how singular a representation is, when restricted to the unipotent radical of a maximal parabolic subgroup.

In Chapter I of this paper, we show that rank is well defined for irreducible unitary representations of GL_n.

We then proceed in Chapter II to classify unitary representations of small rank. If $\operatorname{rank}\rho = k < [n/2]$, then ρ is parabolically induced from the tensor product of an irreducible representation of GL_{n-k} with a character of GL_k. It is worth noting that such a result is false for symplectic groups [H3].

The material in Chapter II is related, in a not readily apparent way, to the theory of reductive dual pairs [H2].

It turns out that the correspondence θ established in §II.3 is the same as the oscillator correspondence (between GL_n and GL_k) when the character χ is fixed to be the trivial character.

The most important results are contained in Chapter III. We give estimates on the asymptotic decay of matrix coefficients of representations, based on their rank. These estimates refine and quantify Theorem 0.1.

Theorem III.3.5 is a generalization of the result of D. A. Kazhdan [K] that the connected component of the trivial representation of GL_n, $n \geq 3$, consists of characters. It states that for k small enough, the set of representations of rank k is both open and closed in GL_n. We obtain as a byproduct a new and quantitative proof of Kazhdan's result.

The estimates we give can be usefully compared with some results of Langlands and Zuckerman [Z] on the asymptotics of matrix coefficients of admissible representations to obtain the nonunitarizability of certain induced $(\mathfrak{g}, K)$-modules [Sa1].

It is possible to compute the rank of a unitary representation in terms of the parameters appearing in the Tadić-Vogan classification, and thus obtain information about specific representations that are given in terms of these parameters.

This fits in better with the methods of a subsequent paper and we shall do it there.

1. Notation

The category of continuous representations of a locally compact group G, with intertwining operators as morphisms, will be denoted by $\operatorname{Rep} G$. The set of equivalence classes of irreducible unitary representations of G will be denoted by $\widehat{G}$. We endow $\widehat{G}$ with the Fell topology [F1].

We shall often omit the adjective "unitary" in the sequel.

We fix a locally compact nondiscrete field F throughout this paper. Let μ be Haar measure on F. We shall denote by $|\ |$ an absolute value on F, normalized in such a way that if E is a measurable subset of F and $x \in F$,

$$\mu(xE) = |x|\mu(E). \tag{1}$$

The group $\mathrm{GL}_n(F)$ will be denoted by G_n.

Let k be an integer satisfying $0 < k < n$. Then $P_k^{(n)}$ will denote the maximal parabolic subgroup of G_n defined by

$$P_k^{(n)} = \{g \in G_n,\ g_{ij} = 0 \text{ if } i > k \text{ and } j \le k\}. \tag{2}$$

The group $P_k^{(n)}$ is the stabilizer of the span of the first k basis vectors in the standard left action of G_n on F^n.

The unipotent radical of $P_k^{(n)}$ will be denoted by $N_k^{(n)}$. Namely

$$N_k^{(n)} = \{g \in G_n,\ g_{ij} = \delta_{ij} \text{ if } i > k \text{ or } j \le k\}. \tag{3}$$

Note that $N_k^{(n)}$ is abelian and naturally isomorphic to $\mathrm{Hom}_F(F^{n-k}, F^k)$.

The Levi component of $P_k^{(n)}$ will be denoted by $L_k^{(n)}$:

$$L_k^{(n)} = P_k^{(n)} \cap (P_k^{(n)})^t, \tag{4}$$

where t denotes the transpose operation on matrices. Note that $L_k^{(n)} \simeq G_k \times G_{n-k}$. Also $P_k^{(n)}$ is a semidirect product

$$P_k^{(n)} = L_k^{(n)} \ltimes N_k^{(n)} \tag{5}$$

and the action by conjugation of $L_k^{(n)}$ on $N_k^{(n)}$ becomes the standard action of $G_k \times G_{n-k}$ on $\mathrm{Hom}_F(F^{n-k}, F^k)$ by pre- and post-multiplication:

$$T \cdot (A, B) = A^{-1}TB. \tag{6}$$

We shall also denote the parabolic subgroup $P_k^{(n)} \cap P_{n-k}^{(n)}$ by $Q_k^{(n)}$ and its unipotent radical by $H_k^{(n)}$.

We shall omit, for all the groups just defined, the superscript ${}^{(n)}$ when no confusion can arise.

2. Rank for representations of $P_k^{(n)}$

We fix, for the rest of this paper, a nontrivial character χ_1 of F. We can now identify $(N_k^{(n)})^\wedge$ with $\mathrm{Hom}_F(F^k, F^{n-k}) = (N_k^{(n)})^*$ as follows: if $n^* \in \mathrm{Hom}(F^k, F^{n-k})$ and $n \in \mathrm{Hom}_F(F^{n-k}, F^k)$, define $\chi_{n^*} \in (N_k^{(n)})^\wedge$ by

$$\chi_{n^*}(n) = \chi_1(\mathrm{tr}(n^* \circ n)). \tag{7}$$

The adjoint action on $(N_k^{(n)})^\wedge$ induced by the action by conjugation of $L_k^{(n)}$ on $N_k^{(n)}$ now corresponds to the standard action of $G_k \times G_{n-k}$ on $\mathrm{Hom}_F(F^k, F^{n-k})$.

In particular the $L_k^{(n)}$-orbits on $(N_k^{(n)})^\wedge$ are

$$\mathcal{O}_l = \{n^*,\ n^* \text{ is a linear transformation of rank } l\} \tag{8}$$

for integers l satisfying $0 < l \le \min(k, n-k)$.

The groups $N_k^{(n)}$ and $P_k^{(n)}$ are well known to be of type I and it is easily seen that the $\mathcal{O}_l$ are locally closed subsets $(N_k^{(n)})^\wedge$. Hence the extension $1 \to N_k^{(n)} \to P_k^{(n)} \to L_k^{(n)} \to 1$ satisfies the hypotheses for Mackey theory [R].

Let $\rho \in \operatorname{Rep} N_k^{(n)}$. Write

$$\rho \simeq \int_{(N_k^{(n)})^*}^{\oplus} m_{n^*} \chi_{n^*}\, d\mu(n^*) \tag{9}$$

for some multiplicity function m and some Borel measure μ on $(N_k^{(n)})^*$.

If $E \subseteq (N_k^{(n)})^*$ is measurable, set $\mu_i(E) = \mu(E \cap \mathcal{O}_i)$. Then

$$\mu = \mu_0 \oplus \mu_1 \oplus \cdots \oplus \mu_{\min(k, n-k)}. \tag{10}$$

Also

$$\rho = \rho_0 \oplus \cdots \oplus \rho_{\min(k, n-k)}, \tag{11}$$

where

$$\rho_i = \int_{(N_k^{(n)})^*}^{\oplus} m_{n^*} \chi_{n^*}\, d\mu_i(n^*). \tag{12}$$

If ρ is actually the restriction of a representation of $P_k^{(n)}$, it is clear that $\rho = \rho_0 \oplus \cdots \oplus \rho_{\min(k, n-k)}$ also as a representation of $P_k^{(n)}$.

Definition 2.1. *A unitary representation* ρ of $P_k^{(n)}$ has pure rank r if $\rho_m = 0$ for all $m \ne r$.

In particular representations of $P_k^{(n)}$ have pure rank.

3. The rank of a representation of GL_n

Definition 3.1. Let $\rho \in \operatorname{Rep} G_n$. We say ρ has pure N_k-rank if $\rho|_{P_k^{(n)}}$ has pure rank r.

Theorem 3.2. *Let $\rho \in \operatorname{Rep} G_n$ and suppose ρ has pure N_k-rank r and pure N_l-rank s. Then*

(1) *If both r and s are $\le \min(k, l, n-k, n-l)$, then $r = s$.*
(2) *If $r > \min(l, n-l)$, then $s = \min(l, n-l)$.*

Corollary 3.3. *Suppose $\rho \in \operatorname{Rep} G_n$ has pure N_k-rank for all k. Then*

$$N_k - \operatorname{rk}(\rho) = \min(k, n-k, N_{[n/2]} - \operatorname{rk}\rho). \quad \square$$

Hence the subgroup $N_{[n/2]}$ detects "all rank there is".

Proof of Theorem 3.2. Let $\phi: A \to B$ be a morphism of abelian groups. Then we have an adjoint map $\hat{\phi}: \operatorname{Rep} B \to \operatorname{Rep} A$ given by $\hat{\phi}(\pi) = \pi \circ \phi$. Note that $\hat{\phi}$ maps $\widehat{B}$ to $\widehat{A}$. If μ is a measure on $\widehat{B}$, $\hat{\phi}_*(\mu)$ is a measure on $\widehat{A}$. If $\pi \in \operatorname{Rep} B$ has spectral measure μ, then $\hat{\phi}(\pi)$ has spectral measure $\hat{\phi}_*(\mu)$.

Let $Z = N_k \cap N_l$. Then Z is a normal subgroup of $P_k \cap P_l$. It can be identified naturally with $\operatorname{Hom}_F(F^{\min(n-k, n-l)}, F^{\min(k, l)})$. As in §1, $\widehat{Z}$ can be identified with $Z^* = \operatorname{Hom}_F(F^{\min(k, l)}, F^{\min(n-k, n-l)})$.

The injections $i_k: Z \to N_k$ and $i_l: Z \to N_l$ induce projections $p_k: N_k^* \to Z^*$ and $p_l: N_l^* \to Z^*$. Under our identifications these correspond to the maps $\hat{i}_k: \widehat{N}_k \to \widehat{Z}$ and $\hat{i}_l: \widehat{N}_l \to \widehat{Z}$ defined above.

The $P_k \cap P_l$ orbits on Z^* are $\mathscr{O}_t'$ for $0 \le t \le \min(k, l, n-k, n-l)$, where

(13) $$\mathscr{O}_t' = \{z^*, \text{ the rank of the linear transformation } z^* \text{ is } t\}.$$

Note that there is only one quasi-invariant measure class on $\mathscr{O}_r^{(k)}$, the P_k-orbit in N_k^* consisting of linear transformations of rank k, namely the class of Lebesgue measure.

Hence the spectral measures μ_k and μ_l of $\rho|_{N_k}$ and $\rho|_{N_l}$ are (equivalent to) respectively Lebesgue measure on $\mathscr{O}_r^{(k)}$ and $\mathscr{O}_s^{(l)}$.

Suppose $r \le \min(k, l, n-k, n-l)$. If $n^* \in \mathscr{O}_r^{(k)}$, then certainly $\operatorname{rk}(p_k n^*) \le r$. Furthermore the condition for $\operatorname{rk}(p_k n^*) < r$ is that n^* satisfy a set of nontrivial polynomial equations. Hence the set $\{n^* \in \mathscr{O}_r^{(k)}, \operatorname{rk}(p_k n^*) < r\}$ has Lebesgue measure zero in $\mathscr{O}_r^{(k)}$. Analogously, if $r > \min(k, l, n-k, n-l)$, almost all $p_k n^*$ lie in $\mathscr{O}'_{\min(k, l, n-k, n-l)}$. We see that

(14)
$$(p_k)_*(\mu_k) = \begin{cases} \text{Lebesgue measure on } \mathscr{O}_r' & \text{if } r \le \min(k, l, n-k, n-l), \\ \text{Lebesgue measure on } \mathscr{O}'_{\min(k, l, n-k, n-l)} & \text{otherwise}, \end{cases}$$

and analogously for $(p_l)_*(\mu_l)$.

The spectral measure $\rho|_Z$ must be (equivalent to) both $(p_k)_*(\mu_k)$ and $(p_l)_*(\mu_l)$. This clearly implies the theorem. □

Definition 3.4. Suppose that $\rho \in \operatorname{Rep} G_n$ has pure N_k-rank for all k. Then the $N_{[n/2]}$-rank of ρ is called the N-rank of ρ and denoted by $\operatorname{rk} \rho$.

Lemma 3.5. *Let $\rho \in \operatorname{Rep} G_n$. Then the following are equivalent.*

(1) *ρ has pure N_k-rank zero for some k.*
(2) *ρ has N-rank zero.*
(3) *ρ is a direct integral of characters of G_n.*

Proof. (3) implies (2) since all characters of G_n have the form $\chi \circ \det$ for some character χ of F^x, the group of nonzero elements of the base field, hence are trivial on $N_{[n/2]}$. (2) implies (1) trivially. It follows easily from Corollary 0.2 that (1) implies (3). □

The preceding lemma is essentially a restatement of Theorem 0.1. It can be seen as a particular case of the following theorem.

Theorem 3.6. *Let $\rho \in \operatorname{Rep} G_n$ and let $\rho|_{P_l} = \rho_0 \oplus \cdots \oplus \rho_{\min(l, n-l)}$, be the decomposition of $\rho|_{P_l}$ into subrepresentations of fixed N_l-rank. Let $\mathscr{H}_k$ be the representation space of ρ_k. Then for every k, $\mathscr{H}_k$ is $\rho(G_n)$-invariant. In particular if ρ is irreducible, ρ has pure N_k-rank. Thus we have the decomposition*

$$\widehat{G}_n = \bigcup_{0 \le k \le [n/2]} (\widehat{G}_n)_k, \tag{15}$$

where $(\widehat{G}_n)_k$ is the set of irreducible representations of G_n of N-rank k.

An essential part of the proof will be a knowledge of the representation theory of the Heisenberg group $H_1^{(n)} = N_1^{(n)} N_{n-1}^{(n)}$. Write $H_1^{(n)} = \{(x, y, z),\ x \in (F^{n-2})^*,\ y \in F^{n-2},\ z \in F\}$ corresponding to

$$H_1^{(n)} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & I & y \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

The center of $H_1^{(n)}$ is $Z = \{(0, 0, z)\}$. We have

Theorem 3.7 (Stone-Von Neumann [VN]). *$\widehat{H}_1^{(n)}$ can be described as follows: If $\rho \in \widehat{H}_1^{(n)}$, Z acts by a character χ. Then*

(1) *If χ is the trivial character, ρ is a character of $H_1^{(n)}/Z \simeq F^{2(n-2)}$.*

(2) *Let χ be a nontrivial character of Z. Then there is exactly one representation ρ_χ in $\widehat{H}_1^{(n)}$ with central character χ. Let $X = \{(x, 0, 0)\}$.*

Then ρ_χ can be realized on $L^2(X)$ by the following formulae:

$$\begin{aligned} (\rho_\chi(x, 0, 0)f)(x') &= f(x + x'), \\ (\rho_\chi(0, y, 0)f)(x') &= \chi(x'y)f(x'), \\ (\rho_\chi(0, 0, z)f)(x') &= \chi(z)f(x'). \quad \square \end{aligned} \tag{16}$$

Proof of Theorem 3.6. We shall prove, by induction on n, that $\mathscr{H}_k$ is $\rho(G_n)$-invariant if $k < \min(l, n-l)$.

If $n = 2$ or 3 the only possible case is $k = 0$. The required result is Lemma 3.5.

Now let $n > 3$. If $k = 0$ we are again done by Lemma 3.5. If $l = 1$ or $n-1$, $k = 0$ is the only possible case, so we may assume $1 < l < n-1$. In this case P_l and $Q_1 = P_1 \cap P_{n-1}$ generate G_n, so that it is sufficient to prove that $\mathscr{H}_k$ is $\rho(Q_1)$-invariant. We have the Levi decomposition

$$Q_1 = M \ltimes H^{(n)}, \tag{17}$$

where $M \simeq F^\times \times G_{n-2} \times F^\times$.

We examine the representation theory of Q_1. The action of Q_1 on $\widehat{Z}$, the dual of the center Z of $H_1^{(n)}$, has exactly two orbits: {trivial character} and $\widehat{Z}$-{trivial character}. The action is the standard action of $G_1 \times G_1$ on $\mathrm{Hom}_F(F, F)$. Fix a nontrivial character χ_1 of Z. Let

$$J = \{g \in Q_1, \, g_{11} = g_{nn}\}. \tag{18}$$

Then $J = \mathrm{stab}_{Q_1}(\chi_1)$ and [R] tells us that $\mathrm{Ind}_J^{Q_1}$ is an equivalence of categories between the category of representations of Q_1 with no Z-invariant vectors and the category $(\mathrm{Rep}\, J)_1$ of representations of J which restrict to a multiple of χ_1 on Z.

Theorem 3.7 tells us there is exactly one irreducible representation ρ_1 of $H_1^{(n)}$ which restricts to a multiple of χ_1 on Z.

It is very easy to check that ρ_1 extends to the representation ω_1 of J defined as follows: Let $M_1 = J \cap M$,

$$M_1 = \{\mathrm{diag}(\alpha, A, \alpha), \; \alpha \in F^\times, A \in G_{n-2}\}. \tag{19}$$

Then

$$J = M_1 \ltimes H_1^{(n)}. \tag{20}$$

Define

$$(\omega_1(\alpha, A, \alpha)f)(x') = |\alpha|^{(2-n)/2} |\det A|^{1/2} f(\alpha^{-1} x' A) \tag{21}$$

and

$$\omega_1 | H_1^{(n)} = \rho_1. \tag{22}$$

Again [R] says that $\_\otimes\omega_1$ is an equivalence of categories between $\mathrm{Rep}\, M_1$ and $(\mathrm{Rep}\, J)_1$.

In conclusion, if $\rho \in \mathrm{Rep}\, G_n$ has no Z-invariant vectors, then $\rho|_Q \simeq \mathrm{Ind}_J^Q \tau \otimes \omega_1$ and the structure of $\rho|_Q$ is the same as the structure of τ. By induction, $\tau|_{G_{n-2}}$ satisfies the theorem. It suffices to show that if $\tau|_{G_{n-2}}$ has pure rank, then ρ also has pure rank and $\mathrm{rk}\, \rho = \mathrm{rk}\, \tau|_{G_{n-2}} + 1$.

To this end we analyze the spectrum of $\rho|_{N_l}$. The group N_l is a normal subgroup of $P_l \cap Q_1$. The action by conjugation of $P_l \cap Q_1$ on N_l induces a map $P_l \cap Q_1 \to G_l \times G_{n-l}$. The image of $P_l \cap Q_1$ is $P_1^{(l)} \times P_{n-l-1}^{(n-l)}$. The group $P_1^{(l)}$ is the stabilizer in G_l of Fe_1 and $P_{n-l-1}^{(n-1)}$ is the stabilizer in G_{n-l} of $V_{n-l-1} = \mathrm{span}\{e_{l+1} \cdots e_{n-1}\}$.

The $(G_l \times G_{n-l})$-orbit $\mathscr{O}_k \subseteq N_l^*$ decomposes under the action of $P_1^{(l)} \times P_{n-l-1}^{(n-l)}$ into the following orbits:

$$\begin{aligned} &\mathscr{O}_{k,-,i} = \{n^*, \, \mathrm{rk}\, n^* = k, \, n^*(e_1) = 0, \, \dim(\mathrm{Im}\, n^* \cap V_{n-l-2}) = i\}, \\ &\mathscr{O}_{k,*,i} = \{n^*, \, \mathrm{rk}\, n^* = k, \, n^*(e_1) \neq 0, \, n^*(e_1) \in V_{n-l-1}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \dim(\mathrm{Im}\, n^* \cap V_{n-l-1}) = i\}, \\ &\mathscr{O}_{k,+,i} = \{n^*, \, \mathrm{rk}\, n^* = k, \, n^*(e_1) \notin V_{n-l-1}, \, \dim(\mathrm{Im}\, n^* \cap V_{n-l-1}) = i\}. \end{aligned} \tag{23}$$

Write τ as a direct integral of irreducible representations of M. Almost all representations appearing in the direct integral decomposition of τ have pure rank equal to that of $\tau|_{G_{n-2}}$ when restricted to G_{n-2}. The functor $\mathrm{Ind}_J^Q \_ \otimes \omega_1$ preserves direct integrals so we an assume τ to be irreducible.

In this case $\tau = \pi \otimes \chi$ where $\pi = \tau|_{G_{n-2}}$ and $\chi \in (M_1/G_{n-2})^\wedge = \widehat{F}^\times$. Let $\sigma = \mathrm{Ind}_J^{Q_1} \pi \otimes \chi \otimes \omega_1$.

Since we can choose representatives for $J\backslash Q_1$ which centralize M_1, the Mackey decomposition theorem [M2] shows

$$\sigma = \pi \otimes \chi \otimes \mathrm{Ind}_J^{Q_1} \omega_1 . \tag{24}$$

The orbit $\mathscr{O}_{k,-,k}$ in N_l^* consists of those characters of N_l which factor to $N_l \cap G_{n-2}$. Hence the N_l-spectral measure of π is concentrated on $\mathscr{O}_{\mathrm{rk}\,\pi,-,\mathrm{rk}\,\pi}$.

We next want to show that $\mathrm{Ind}_J^{Q_1} \omega_1$ has spectral measure concentrated on $\mathscr{O}_{1,+,0}$. Indeed, first consider ω_1. If $g \in N_l$, write g as

$$g = \left[\begin{array}{c|c} I & \begin{array}{c|c} x & z \\ \hline B & y \end{array} \\ \hline 0 & I \end{array} \right] . \tag{25}$$

If $x' \in X$, write $x' = (x_1, x_2)$ with $x_1 \in (F^{l-1})^*$, $x_2 \in (F^{n-l-1})^*$. Then, with obvious notation,

$$(\omega_1(x, y, z, B)f)(x_1, x_2) = \chi_1(z)\chi_1(x_2, y)f(x, x_1B + x_2 + x). \tag{26}$$

Let U be the operator consisting of the Fourier transform with respect to x_2. Precisely, if ξ denotes an $(n-l-1)$-column vector, then

$$(Uf)(x_1, \xi) = \int_{F^{n-l-1}} f(x_1, x_2)\chi_1(-x_2\xi)\, dx_2 . \tag{27}$$

Then it is easy to compute that

$$(U\omega_1(x, y, z, B)U^{-1}\phi)(x_1, \xi) = \chi_1(z)\chi_1(x_1y)\chi_1(x_1B\xi)\chi_1(x\xi)\phi(x_1, \xi). \tag{28}$$

We have just written ω_1 as a direct integral

$$\omega_1 \simeq \int_{F^{l-1}\times F^{n-l-1}}^{\otimes} \chi_{x_1,\xi}\, d\mu(x_1, \xi), \tag{29}$$

where

$$\chi_{x_1,\xi}(x, y, z, B) = \chi_1(z)\chi_1(x_1y)\chi_1(x_1B\xi)\chi_1(x\xi). \tag{30}$$

Under our identifications $\chi_{x_1,\xi}$ corresponds to the matrix $\left[\begin{smallmatrix} x_1 & 1 \\ \xi x_1 & \xi \end{smallmatrix}\right]$ which lies in $\mathscr{O}_{1,+,0}$.

We now turn to the effect of $\mathrm{Ind}_J^{Q_1}$ on the N_l-spectrum of a representation. Let $\nu \in \mathrm{Rep}\, J$ and let

$$\nu|_{N_l} \simeq \int_{N_l^*}^{\oplus} m_{n^*}\chi_{n^*}\, d\mu(n^*). \tag{31}$$

We have

$$J\backslash Q_1 \simeq \frac{F^\times \times F^\times}{\mathrm{diag}}. \tag{32}$$

If $(\alpha, \beta) \in F^\times \times F^\times$, set

$$n^* \cdot (\alpha, \beta) = \begin{bmatrix} \alpha & 0 \\ 0 & I \end{bmatrix}^{-1} n^* \begin{bmatrix} I & 0 \\ 0 & \beta \end{bmatrix}. \tag{33}$$

Let $\mu \cdot (\alpha, \beta)$ and $m_{n^*} \cdot (\alpha, \beta)$ be induced by this action. Let $d(\overline{\alpha, \beta})$ be Lebesgue measure on $(F^\times \times F^\times)/\mathrm{diag}$. Then Mackey decomposition [M2] tells us that

$$(\mathrm{Ind}_J^{Q_1} \nu)|_{N_l} = \int_{(F^\times \times F^\times)/\mathrm{diag}}^{\oplus} \int_{N_l^*}^{\oplus} m_{n^*} \cdot (\alpha, \beta) \chi_{n^*} d(\mu \cdot (\alpha, \beta))(n^*) d(\overline{\alpha, \beta}). \tag{34}$$

This discussion, applied to $\nu = \omega_1$, shows that the N_l-spectral measure of $\mathrm{Ind}_J^{Q_1} \omega_1$ is concentrated on $\mathscr{O}_{1,+,0}$.

Finally we show that the spectral measure of $\sigma = \pi \otimes \chi \otimes \mathrm{Ind}_J^Q \omega$ is concentrated on $\mathscr{O}_{\mathrm{rk}\,\pi+1}$. Indeed $\mathscr{O}_{1,+,0} + \mathscr{O}_{\mathrm{rk}\,\pi,-,\mathrm{rk}\,\pi}$ is Zariski open (hence of total measure) in $\mathscr{O}_{\mathrm{rk}\,\pi+1}$.

Let π' and π'' be two representations of N_l. Consider the (outer) tensor product $\pi' \otimes \pi''$ of π' and π''. This is a representation of $N_l \times N_l$. Its spectrum is the Cartesian product of the spectra of π' and π''. Taking the inner tensor product amounts to taking the restriction to the diagonal of the outer tensor product. If $\mathrm{diag}: N_l \to N_l \times N_l$, let $\mathrm{diag}^*: N_l^* \times N_l^* \to N_l^*$ be the adjoint map.

We have

$$\mathrm{diag}^*(n_1^*, n_2^*) = n_1^* + n_2^*. \tag{35}$$

Hence

$$\mathrm{diag}^*(\mathscr{O}_{1,+,0} \times \mathscr{O}_{\mathrm{rk}\,\pi,-,\mathrm{rk}\,\pi}) = \mathscr{O}_{1,+,0} + \mathscr{O}_{\mathrm{rk}\,\pi,-,\mathrm{rk}\,\pi}. \tag{36}$$

This concludes the proof of Theorem 3.6. □

II. CLASSIFICATION OF REPRESENTATIONS OF SMALL RANK

1. Irreducibility properties

The main result of this section is

Theorem 1.1. *Let $\rho, \rho' \in (\widehat{G}_n)_k$, $k < [n/2]$, and $l \geq k$. Then $\mathrm{Hom}_{G_n}(\rho, \rho') = \mathrm{Hom}_{Q_l(n)}(\rho, \rho')$. In particular $\rho|_{Q_l}$ is irreducible and if $\rho \simeq \rho'$, then $\rho|_{Q_l} \simeq \rho'|_{Q_l}$, so that restriction to Q_l defines an injection $(\widehat{G}_n)_k \to \widehat{Q}_l^{(n)}$.*

Before passing to the proof, we wish to discuss a certain representation of G_2. We define $\omega_1^{(2)} \in \mathrm{Rep}\, G_2$ as follows: Its representation space is $L^2(F^2)$ and if $g \in G_2$, $f \in L^2(F^2)$ and $x \in F^2$, we define

$$(\omega_1^{(2)}(g)f)(x) = |\det g|^{1/2} f(xg). \tag{1}$$

Lemma 1.2. (a) $\operatorname{Hom}_{G_2}(\omega_1^{(2)}, \omega_1^{(2)}) = \operatorname{Hom}_{Q_1^{(2)}}(\omega_1^{(2)}, \omega_1^{(2)})$.

(b) *Let* λ *and* λ' *be characters of* G_2. *Then if* $\lambda \neq \lambda'$, *the intersection of the spectra of* $\lambda \otimes \omega_1^{(2)}$ *and* $\lambda' \otimes \omega_1^{(2)}$ *has zero measure in both spectra. In particular* $\operatorname{Hom}_{G_2}(\lambda \otimes \omega_1^{(2)}, \lambda' \otimes \omega_1^{(2)}) = \{0\}$.

Proof. For the purposes of this proof we abbreviate G_2 to G, $Q_1^{(2)}$ to Q, and $\omega_1^{(2)}$ to ω. Let N be the unipotent radical of Q and A its Levi component. Also, let Z denote the center of G. If we apply the Mackey theory of representations of group extensions [R] to $Q = A \ltimes N$ we obtain that the irreducible representations of Q are:

(a) characters of A;

(b) $\operatorname{Ind}_{ZN}^{Q} \chi \otimes \chi_1$, $\chi \in \widehat{Z}$, χ_1 a fixed character of N.

In particular, if $\rho \in \operatorname{Rep} Q$ and $\rho|_N$ is equivalent to the regular representation of N, then ρ is irreducible.

Also note that

$$(\operatorname{Ind}_{ZN}^{Q} \chi \otimes \chi_1)|_Z = \infty \chi . \tag{2}$$

As a measure space, $F^2 \simeq F^2 - \{0\}$. G acts transitively on $F^2 - \{0\}$. The stabilizer of $(0, 1)$ is

$$S = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}, x \in F^\times, y \in F \right\}. \tag{3}$$

Hence, as homogeneous spaces, $F^2 - \{0\} \simeq S \backslash G$ and ω can be viewed as the regular representation of G on $L^2(S\backslash G)$. We have

$$\omega = \operatorname{Ind}_S^G 1_S , \tag{4}$$

where 1_S denotes the trivial representation of S. We break up the induction in two stages:

$$\omega = \operatorname{Ind}_Q^G \operatorname{Ind}_S^Q 1_S . \tag{5}$$

Let

$$G' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}, \ x \in F^\times \right\} \simeq F^\times . \tag{6}$$

Then it is easy to see that

$$\operatorname{Ind}_S^Q 1_S = 1_S \otimes \operatorname{regular}_{G'} . \tag{7}$$

The regular representation of G' breaks up as a direct integral over all characters. Since induction commutes with direct integrals, we have

$$\omega = \int_{\widehat{F^\times}}^{\oplus} \operatorname{Ind}_Q^G (1_S \otimes \chi)\, d\chi . \tag{8}$$

To prove (a) it will be enough to show that $\operatorname{Ind}_Q^G 1_S \otimes \chi$ is irreducible when restricted to Q (this will also give the spectral decomposition of ω) and that the $\operatorname{Ind}_Q^G 1_S \otimes \chi$ are distinct when restricted to Q.

Let $w = [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}]$. Then the double coset QwQ has total measure in $Q\backslash G/Q$. Note that $Q^w \cap Q = A$.

The Mackey decomposition theorem [M2] says that

$$(9) \qquad (\mathrm{Ind}_Q^G 1_S \otimes \chi)|_Q = \mathrm{Ind}_A^Q (1_S \otimes \chi)^w |_A.$$

We have

$$(10) \qquad (1_S \otimes \chi)^w|_A = \chi \otimes 1_{G'},$$

and it is clear (for example from Mackey decomposition again) that $(\mathrm{Ind}_A^Q \chi \otimes 1_{G'})|_N$ is equivalent to the regular representation of N. Hence $\mathrm{Ind}_Q^G \chi \otimes 1_{G'}$ is irreducible. Finally

$$(11) \qquad (\mathrm{Ind}_A^Q \chi \otimes 1_{G'}) \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = \chi(x) I$$

so that already the $(\mathrm{Ind}_A^Q \chi \otimes 1_{G'})|_Z$ are distinct. Statement (a) is proved.

As for (b), as remarked earlier, (8) gives the spectral decomposition of ω. If λ is a character of G, then λ is of the form $\tilde{\lambda} \circ \det$ for some $\tilde{\lambda} \in \widehat{F}^\times$. It is then clear that

$$(12) \qquad \lambda \otimes \mathrm{Ind}_Q^G 1_S \otimes \chi = \mathrm{Ind}_Q^G \tilde{\lambda} \otimes \tilde{\lambda}\chi.$$

It is well known that if $\alpha, \beta, \gamma, \delta \in \widehat{F}^\times$, then $\mathrm{Ind}_Q^G \alpha \otimes \beta \simeq \mathrm{Ind}_Q^G \gamma \otimes \delta$ if and only if $\alpha = \gamma$ and $\beta = \delta$ or $\alpha = \delta$ and $\beta = \gamma$ [W, BZ].

Hence the intersection of the spectra of $\lambda \otimes \delta$ and $\lambda' \otimes \omega$ consists of only one point and Lemma 1.2 is proved. □

We now pass to the proof of Theorem 1.1. We have to show that any Q_l-morphism of ρ and ρ' also intertwines the G_n-action. If $k = 0$, ρ and ρ' are characters (Lemma I.3.5) and the conclusion of the theorem is obvious, since characters of G_n are all distinct on Q_l for any l.

From now on, suppose $k > 0$.

We shall use the notation developed in the proof of Theorem I.3.6. We shall also write

$$(13) \qquad \tau = \mathrm{Ind}_J^Q \omega_1,$$

$$(14) \qquad A = \{g \in J,\ g_{ij} = \delta_{ij} \text{ if } (i, j) \neq (1, 1) \text{ or } (n, n)\}.$$

The proof of Theorem I.3.6 shows that under our assumptions on the rank of ρ,

$$(15) \qquad \rho|_{Q_1} = \tau \otimes \int_{\widehat{F}^\times}^{\oplus} \chi \otimes \nu_\chi \, d\mu(\chi),$$

where the χ's are considered as characters of A, $\nu_\chi \in \mathrm{Rep}\, G_{n-2}$, ,and

$$(16) \qquad \mathrm{rk}\, \nu_\chi = \mathrm{rk}\, \rho - 1.$$

We can also write

$$\rho'|_{Q_1} = \tau \otimes \int_{\widehat{F}^\times}^{\oplus} \chi \otimes \nu'_\chi \, d\mu'(\chi) \tag{17}$$

with the analogous stipulations.

Suppose first $k = l = 1$. This forces $n \geq 4$. Also $\operatorname{rk} \nu_\chi = 0$ so that ν_χ is a direct integral of characters. Hence we can write

$$\rho|_{Q_1} = \tau \otimes \int_{(\widehat{G}_{n-2})_0 \otimes \widehat{F}^\otimes}^{\oplus} n_{\lambda,\chi} \lambda \otimes \chi \, d\tilde{\mu}(\lambda, \chi). \tag{18}$$

I claim that $\tilde{\mu}$ is actually concentrated on a single line $\lambda =$ constant. Indeed, write the representation space $\mathscr{H}$ of ρ as

$$\mathscr{H} = \int_{(\widehat{G}_{n-2})_0}^{\oplus} \mathscr{H}_\lambda \, d\tilde{\tilde{\mu}}(\lambda) \tag{19}$$

corresponding to the decomposition of $(\widehat{G}_{n-2})_0 \times F^\times$ along the lines $\lambda =$ constant. This decomposition is Q_1-invariant.

Let

$$\widetilde{G}_2 = \{g \in G_n,\ g_{ij} = \delta_{ij} \text{ unless } (i, j) = (1, 1), (1, n), (n, 1), (n, n)\}. \tag{20}$$

Note that $\widetilde{G}_2$ commutes with G_{n-2}. We claim that the G_{n-2}-spectra of the representations ρ_λ on the $\mathscr{H}_\lambda$ are disjoint (apart from sets of zero measure). This will be proved below.

Hence the decomposition (19) is also $\widetilde{G}_2$-invariant. Since $\widetilde{G}_2$ and Q_1 generate G_n, the decomposition is G_n-invariant. But we are supposing ρ irreducible. Hence $\mathscr{H} = \mathscr{H}_\lambda$ for some λ and $\tilde{\mu}$ is concentrated on one line.

It is easy to see, for example from the Mackey decomposition theorem [M2], that, since we can choose representatives for $J \backslash Q_1$ that centralize G_{n-2},

$$\tau|_{G_{n-2}} = \infty \omega_1. \tag{21}$$

Hence

$$\rho|_{G_{n-2}} = \infty \lambda \otimes \omega_1 \tag{22}$$

with $\lambda \in (\widehat{G}_{n-2})_0$. Analogously,

$$\rho'|_{G_{n-2}} = \infty \lambda' \otimes \omega_1. \tag{23}$$

If $\lambda \neq \lambda'$, by our previous claim that the G_{n-2}-spectra of $\lambda \otimes \omega_1$ and $\lambda' \otimes \omega_1$ are disjoint, $\operatorname{Hom}_{G_{n-2}}(\rho, \rho') = \{0\}$. Hence $\operatorname{Hom}_{Q_1}(\rho, \rho') = \{0\}$ and there is nothing to prove. So from now on, suppose that $\lambda = \lambda'$. Let

$$Q_1^{(2)} = Q_1^{(n)} \cap \widetilde{G}_2. \tag{24}$$

Since $\widetilde{G}_2$ and $Q_1^{(n)}$ generate Q_n, it will be enough to show that

$$\operatorname{Hom}_{Q_1^{(n)}}(\rho, \rho') \subseteq \operatorname{Hom}_{\widetilde{G}_2}(\rho, \rho'),$$

or even $\operatorname{Hom}_{Q_1^{(2)}}(\rho, \rho') \subseteq \operatorname{Hom}_{\widetilde{G}_2}(\rho, \rho')$.

Since $n \geq 4$ we can conjugate $\widetilde{G}_2$ into a subgroup of G_{n-2} by an element of G_n. Let $\omega_1^{(2)}$ be as defined in (1). Then it is clear from (I.21) that

$$\omega_1|_{\widetilde{G}_2} = \infty\omega_1^{(2)}. \tag{25}$$

Also, $\lambda \in (\widehat{G}_{n-2})_0$ is of the form $\tilde{\lambda} \circ \det$ for some $\tilde{\lambda} \in \widehat{F}^\times$. Then $\lambda|_{\widetilde{G}_2} = \tilde{\lambda} \circ \det$ for the same $\tilde{\lambda}$. We conclude that $\rho|_{\widetilde{G}_2} = \infty\lambda \otimes \omega_1^{(2)} = \rho'|_{\widetilde{G}_2}$.

Now Lemma 1.2(a) implies $\mathrm{Hom}_{Q_1^{(2)}}(\rho, \rho') = \mathrm{Hom}_{\widetilde{G}_2}(\rho, \rho')$, which is what we had to show.

To conclude the proof of the case $k = l = 1$ we must verify our claim that the G_{n-2}-spectra of the ρ_λ are distinct.

We know that $\rho_\lambda|_{G_{n-2}} = \infty\lambda \otimes \omega_1$ (equation (22)). Since the spectra of $\infty\lambda \otimes \omega_1|_{\widetilde{G}_2}$ and $\infty\lambda \otimes \omega_1|_{\widetilde{G}_2}$ are already disjoint ((25) and Lemma 1.2(b)), the claim is true.

Finally, we prove by induction the general case. Recall that we had the decomposition (15) with $\mathrm{rk}\,\nu_\chi = \mathrm{rk}\,\rho - 1$. By induction on n we can assume that

$$\mathrm{Hom}_{Q_{l-1}^{(n-2)}}(\nu_\chi, \nu'_\chi) = \mathrm{Hom}_{G_{n-2}}(\nu_\chi, \nu'_\chi). \tag{26}$$

Consider the group $M = (F^\times \times F^\times)H_1$. It acts on $\widehat{H}_1$ by conjugation and we have

$$\mathrm{stab}_M(\rho_1) = J \cap M. \tag{27}$$

The representation ρ_1 can be extended to $J \cap M$ via the restriction to $J \cap M$ of the representation ω_1 of J.

Theorem 8.4 of [R] implies that for $\chi \in \widehat{A}$, $\mathrm{Ind}_{J\cap M}^M \chi \otimes \omega_1 = \chi \otimes \mathrm{Ind}_{J\cap M}^M \omega_1$ is irreducible. Observe that since $J \cap M\backslash M = J\backslash Q_1^{(n)}$, we have $\mathrm{Ind}_{J\cap M}^M \omega_1 = (\mathrm{Ind}_J^{Q_1} \omega_1)|_M = \tau|_M$. We can obviously extend $\chi \otimes \tau|_M$ to the representation $\chi \otimes \tau$ of $Q_1^{(n)}$. Since M is normal in $Q_1^{(n)}$, Theorem 8.4 of [R] says that $\tau \otimes \chi \otimes \_$ is an equivalence of categories between the category of representations of $Q_1^{(n)}$ which restrict to a multiple of $\chi \otimes \tau|_M$ and $\mathrm{Rep}(Q_1^{(n)}/M) = \mathrm{Rep}\,G_{n-2}$.

In other words

$$\mathrm{Hom}_{Q_1}(\tau \otimes \chi \otimes \nu_\chi, \tau \otimes \chi \otimes \nu'_\chi) = \mathrm{Hom}_{G_{n-2}}(\nu_\chi, \nu'_\chi). \tag{28}$$

For the same reason,

$$\mathrm{Hom}_{Q_1 \cap Q_l}(\tau \otimes \chi \otimes \nu'_\chi, \tau \otimes \chi \otimes \nu'_\chi) = \mathrm{Hom}_{Q_{l-1}^{(n-2)}}(\nu_\chi, \nu'_\chi). \tag{29}$$

Hence by (26)

$$\mathrm{Hom}_{Q_l \cap Q_1}(\rho, \rho') = \mathrm{Hom}_{Q_1}(\rho, \rho'). \tag{30}$$

Let $T \in \mathrm{Hom}_{Q_l}(\rho, \rho') \subseteq \mathrm{Hom}_{Q_l \cap Q_1}(\rho, \rho')$. Then $T \in \mathrm{Hom}_{Q_1}(\rho, \rho')$ so that $T \in \mathrm{Hom}_{Q_1}(\rho, \rho') \cap \mathrm{Hom}_{Q_l}(\rho, \rho')$. Since Q_1 and Q_l generate G_n, $T \in \mathrm{Hom}_{G_n}(\rho, \rho')$. This concludes the proof of Theorem 2.1. □

2. The map θ

We have seen that restriction defines an injection $(\widehat{G}_n)_k \to \widehat{G}_l^{(n)}$ if $l \geq k$. In fact much more is true, as we shall see in this section and the next.

Let $\rho \in (\widehat{G}_n)_k$ and $k < [n/2]$. Then $\rho|_{Q_k}$ is still irreducible (Theorem 2.1).

We examine the representation theory of Q_k. The center Z of the unipotent radical H_k of Q_k is isomorphic to $\mathscr{M}_k$, the additive group of $k \times k$ matrices. Fix a nontrivial character χ_1 of F. Any character of $\mathscr{M}_k$ is of the form $\chi_1 \circ \lambda$ for some $\lambda \in \mathscr{M}_k^*$. We identify $\mathscr{M}_k^*$ with $\mathscr{M}_k$ by setting, for $A, B \in \mathscr{M}_k$,

$$\lambda_A(B) = \mathrm{trace}(AB). \tag{31}$$

The action of Q_k on $\widehat{\mathscr{M}_k}$ induced by conjugation becomes, under these identifications, the usual action of $G_k \times G_k$ on $\mathscr{M}_k$ by pre- and post-multiplication.

Hence orbits are, for $0 \leq l \leq k$,

$$\mathscr{O}_l' = \{\lambda_A, \ \mathrm{rank}(A) = l\}. \tag{32}$$

If $\lambda \in \mathscr{M}_k^*$, $\ker \lambda$ is a normal subgroup of $H_k^{(n)}$. If λ has rank k, $H_k/\ker \lambda$ is a Heisenberg group with center $\mathscr{M}_k/\ker \lambda$. Let $\sigma \in \widehat{H}_k$. Then $\sigma|_Z$ is a multiple of a character. If this character happens to correspond to a functional λ of rank k, σ factors to $H_k/\ker \lambda$. The Stone-Von Neumann theorem [VN] says that there is a unique irreducible representation of $H_k/\ker \lambda$ with any fixed nontrivial central character. Hence, for any $\lambda \in \mathscr{O}_k'$, there is a unique irreducible representation ρ_λ of H_k which restricts to $\chi_1 \circ \lambda$ on Z.

If we write

$$H_k = \{(x, y, z), \ x \in \mathscr{M}_{k, n-2k}, \ y \in \mathscr{M}_{n-2k, k}, \ z \in \mathscr{M}_k\} \tag{33}$$

corresponding to the block decomposition

$$\begin{bmatrix} I & x & z \\ 0 & I & y \\ 0 & 0 & I \end{bmatrix}, \tag{34}$$

and we set $X = \{(x, 0, 0)\}$, then ρ_λ is realized on $L^2(X)$ by

$$\begin{aligned} (\rho_\lambda(x, 0, 0)f)(x') &= f(x + x'), \\ (\rho_\lambda(0, y, 0)f)(x') &= \chi_1(\lambda(x'y))f(x'), \\ (\rho_\lambda(0, 0, z)f)(x') &= \chi_1(\lambda(z))f(x'). \end{aligned} \tag{35}$$

Let $i: \mathscr{M}_k \to N_k$ be the natural injection of abelian groups. If $\rho \in \mathrm{Rep}\, N_{k'}$ the $\mathscr{M}_k$-spectral measure of ρ is the image of the N_k-spectral measure of ρ under the adjoint projection $i^*: N_k^* \to \mathscr{M}_k^*$. If $\rho \in (\widehat{G}_n)_k$, its N_k-spectral measure is Lebesgue measure on the orbit $\mathscr{O}_k \subseteq N_k^*$.

Hence ρ obviously has $\mathscr{M}_k$-spectral measure concentrated on $\mathscr{O}_k'$. The trace functional on $\mathscr{M}_k$ has rank k. Denote it by tr. Q_k acts on $\widehat{H}_k$ by conjugation. Note that

$$(36)\qquad \begin{aligned} \operatorname{stab}_{Q_k}(\rho_{\mathrm{tr}}) &= \operatorname{stab}_{Q_k}(\mathrm{tr}) \\ &= J = \{g \in Q_k,\ g_{ij} = g_{i+n-k,\, j+n-k} \text{ for all } 1 \le i,\ j \le k\}. \end{aligned}$$

Let

$$(37)\qquad \widehat{J}_{\mathrm{tr}} = \{\tau \in \widehat{J},\ \tau|_{H_k} \text{ is a multiple of } \rho_{\mathrm{tr}}\}.$$

Theorem 1.1 of [R] says that $\operatorname{Ind}_J^{Q_k}$ is an equivalence of categories between the category of representations of Q_k which live on $\mathscr{O}_k'$ and the category of representations of J which restrict to a multiple of ρ_{tr} on H_k.

Since $\rho|_{Q_k}$ is irreducible, $\rho|_{Q_k} = \operatorname{Ind}_j^{Q_k} \tau$ for some $\tau \in \widehat{J}_{\mathrm{tr}}$.

One can easily check that ρ_{tr} extends to J in the following way. The Levi component of J is the group

$$(38)\qquad M = \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & A \end{bmatrix},\ A \in G_k,\ B \in G_{n-2k} \right\}.$$

The representation ω_{tr} of M acts on $L^2(X)$ and is given by

$$(39)\qquad \left(\omega_{\mathrm{tr}} \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & A \end{bmatrix} f \right)(x') = |\det A|^{(2k-n)/2} |\det B|^{k/2} f(A^{-1}x'B).$$

We shall denote by ω_{tr} the representation of J which is ρ_{tr} on H_k and ω_{tr} on M.

Again [R, Theorem 8.4] tells us that $\rho|_{Q_k} = \operatorname{Ind}_J^{Q_k} \omega_{\mathrm{tr}} \otimes \nu$ with $\nu \in \widehat{M} = (G_k \times G_{n-2})^\wedge$.

Hence ν is of the form $\nu = \sigma \otimes \nu'$ with $\sigma \in \widehat{G}_k$, $\nu' \in \widehat{G}_{n-2k}$. The same argument as in the proof of Theorem I.3.6 now shows that

$$(40)\qquad \operatorname{rk} \nu' = \operatorname{rk} \rho - k = 0,$$

so that ν' is actually a character, hence of the form $\chi \circ \det$ for some $\chi \in \widehat{F}^\times$.

Thus if $\rho \in (\widehat{G}_n)_k$,

$$(41)\qquad \rho|_{Q_k} = \operatorname{Ind}_J^{Q_k} \omega_{\mathrm{tr}} \otimes \sigma \otimes \chi \circ \det.$$

Theorem 2.1 says that $\rho \mapsto \rho|_{Q_k}$ is an injection and Mackey theory says that $\rho|_{Q_k} \mapsto (\sigma, \chi)$ is an injection.

We summarize the discussion as follows:

Theorem 2.1. *Let* $k < [n/2]$. *There is an injective map* $\theta\colon (\widehat{G}_n)_k \to \widehat{G}_k \times \widehat{F}^\times$ *such that if* $\theta(\rho) = (\sigma, \chi)$, $\rho|_{Q_k} = \operatorname{Ind}_J^{Q_k} \omega_{\mathrm{tr}} \otimes \sigma \otimes \chi \circ \det$.

3. Classification of small representations

We shall now show that the map θ constructed above is a surjection and give an explicit construction for the inverse of θ. This will give a classification of representations of small rank.

As a byproduct, we shall obtain the irreducibility of certain induced representations of G_n.

To simplify notation, whenever $\chi \in \widehat{F}^\times$, we shall denote the character $\chi \circ \det$ of G_n simply by χ. If χ is a character of G_n and $\rho \in \operatorname{Rep} G_n$, we denote by $\chi\rho$ the representation $\chi \otimes \rho$ of G_n. Finally if $\rho \in \operatorname{Rep} G_k$ and $\sigma \in \operatorname{Rep} G_l$, we denote by $\rho \times \sigma$ the representation of G_{k+l} obtained by extending the representation $\rho \otimes \sigma$ of $G_k \times G_l$ trivially to $P_k^{(k+l)}$ and then inducing up to G_{k+l}.

With this notation we have

Theorem 3.1. *Let* $k < [n/2]$, $\sigma \in \widehat{G}_k$, $\chi \in \widehat{F}^\times$. *Consider* χ *as a character of* G_{n-k}. *Then* $\chi^{-1}\sigma \times \chi \in (\widehat{G}_n)_k$ *and the maps* θ *and* $(\sigma, \chi) \to \chi^{-1}\sigma \times \chi$ *are inverses.*

Corollary 3.2. $(\widehat{G}_n)_k = \{\tau \times \chi,\ \tau \in \widehat{G}_k,\ \chi \in (\widehat{G}_{n-k})_0\}$.

Corollary 3.3. *The representations* $\tau \times \chi$ *of* G_{k+l}, *with* $\tau \in \widehat{G}_k$ *and* $\chi \in (\widehat{G}_l)_0$, *are irreducible whenever* $l \geq k$.

Note. A slightly more general result than Corollary 3.3 was proved, by completely different methods, by S. Sahi [Sa1]. The irreducibility of *all* representations of the form $\tau \times \sigma$, with τ, σ, unitary would follow from Kirillov's conjecture that all irreducible unitary representations of G_n stay irreducible when restricted to $P_1^{(n)}$. The conjecture has been shown to be true for F non-Archimedean by J. Bernstein [B] and for $F = \mathbf{C}$ by S. Sahi [Sa2].

Proof of Theorem 3.1. Because of Theorems 1.1 and 2.1 it is sufficient to show:

(1) $\chi^{-1}\sigma \times \chi$ has pure rank k.

(2) $\chi^{-1}\sigma \times \chi|_{Q_k} \simeq \operatorname{Ind}_J^{Q_k} \omega_{\mathrm{tr}} \otimes \sigma \otimes \chi$.

In fact, it suffices to prove (2) because $\operatorname{Ind}_J^{Q_k} \omega_{\mathrm{tr}} \otimes \sigma \otimes \chi$ has pure N_l-rank k if $k < l \leq [n/2]$. This follows from equation (40) in §3. Statement (1) can also be proved directly by means of the Mackey decomposition theorem [M2]. We omit the details.

We begin the proof of (2) by applying the Mackey decomposition theorem to the groups G_n, P_k, Q_k.

There is one double coset of total measure in $P_k \backslash G_n / Q_k$, a coset representative of which is the element w of G_n given by

$$w_{ij} = \delta_{i,n+1-j}. \tag{42}$$

We denote by M the group

$$M = P_k^w \cap Q_k = \left\{ \begin{array}{c} \begin{bmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \\ k \end{array} \right\}. \tag{43}$$

Here $*$ denotes any matrix of the right size, subject to the stipulation that the element of $\mathscr{M}_n$ which we obtain is invertible.

The representation $(\chi^{-1}\sigma \otimes \chi)^w$ is a representation of M as in the diagram

$$\begin{bmatrix} \chi & \chi & \\ & \chi & \\ & & \chi^{-1}\sigma \end{bmatrix}. \tag{44}$$

Hence we must prove

$$\operatorname{Ind}_M^{Q_k}(\chi \otimes \chi^{-1}\sigma) \simeq \operatorname{Ind}_J^{Q_k} \omega_{\mathrm{tr}} \otimes \sigma \otimes \chi. \tag{45}$$

We shall do this by finding explicitly an intertwining operator between the two representations.

Note that

$$Q_k = M \ltimes N_{n-k}. \tag{46}$$

In the rest of the proof we shall denote Q_k by Q and N_{n-k} by N. Also denote $\chi \otimes \chi^{-1}\sigma \in \operatorname{Rep} M$ by $\overline{\sigma}$ and $\operatorname{Ind}_M^Q \overline{\sigma}$ by π.

It is clear that restriction to N defines a unitary bijection between the representation space $\mathscr{H}_\pi$ of π and $L^2(N; \mathscr{H}_\sigma)$. Elementary computations show that in this realization of π on $L^2(N; \mathscr{H}_\sigma)$

$$(\pi(mu)f)(u_0) = \overline{\sigma}(m) f((u_0 \cdot m)u)\Delta_N(m)^{1/2}. \tag{47}$$

Here $u_0 \cdot m = m^{-1}um$ and Δ_N denotes the modular function for the action of M on N.

Let now $\mathscr{F}: L^2(N; \mathscr{H}_\sigma) \to L^2(N^*; \mathscr{H}_\sigma)$ be given by Fourier transform. In other words

$$(\mathscr{F}f)(v) = \int_N \chi_1(\langle u, v\rangle) f(u)\, du; \tag{48}$$

then $\pi' = \mathscr{F}\pi\mathscr{F}^{-1}$ is equivalent to π and

$$(\pi'(mu)f)(v) = \Delta_N(m)^{1/2}\overline{\sigma}(m)\chi_1(\langle u^{-1} \cdot m^{-1}, v\rangle) f(v \cdot {}^t m^{-1}). \tag{49}$$

Here $v \cdot {}^t m$ denotes the transpose action of M on N^*.

Let us now turn to the other representation involved. Let $\zeta = \omega_{\mathrm{tr}} \otimes \sigma \otimes \chi \in \hat{J}$ and $\nu = \operatorname{Ind}_J^Q \zeta$.

The representation space $\mathscr{H}_\nu$ of ν is the space of functions $f: Q \to \mathscr{H}_\zeta$ which satisfy

(1) for all $j \in J$ and $q \in Q$, $f(jq) = \zeta(j)f(q)$;

(2) the function $q \mapsto \|f(q)\|$ is in $L^2(J\backslash Q, d\mu)$,

where μ is some quasi-invariant measure on $J\backslash Q$. Set

$$\lambda(q_0, q) = (d(\mu \cdot q)/d\mu)(Jq_0).$$

Then the action of Q on $\mathscr{H}_\nu$ is given by

$$(\nu(q)f)(q_0) = \lambda(q_0, q)^{1/2} f(q_0 q). \tag{50}$$

We fix μ as follows: If

$$q = \begin{bmatrix} W & X & R \\ 0 & Y & S \\ 0 & 0 & Z \end{bmatrix}, \tag{51}$$

set

$$s(Jq) = \begin{bmatrix} ZW^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{52}$$

We take μ to be the preimage under s of Lebesgue measure on $\mathscr{M}_k$.

It is easy to see that if q is given by (51), then

$$\lambda(q_0, q) = |\det Z|^{-k} |\det W|^k. \tag{53}$$

We shall denote $\lambda(q_0, q)$ by $\Delta'(q)$. Let $U' : \mathscr{H}_\nu \to L^2(J\backslash Q, d\mu; \mathscr{H}_\zeta)$ be given by restriction to $s(J\backslash Q)$ followed by s^*. U' is a unitary bijection, so that if $\nu' = U'\nu U'^{-1}$, ν' is equivalent to ν. If $\overline{q}_0 = Jq_0$, define

$$j(\overline{q}_0, q_0) = s(\overline{q}_0) q s(\overline{q_0 q})^{-1}. \tag{54}$$

Then it is easy to compute

$$\nu'(q) f(\overline{q}_0) = \Delta'(q)^{1/2} \zeta(j(\overline{q}_0, q)) f(\overline{q_0 q}). \tag{55}$$

Let

$$\overline{q}_0 = \overline{\begin{bmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}}.$$

Then

$$j(\overline{q}_0, m) = \begin{bmatrix} Z & AX & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{bmatrix} \quad \text{if } m = \begin{bmatrix} W & X & 0 \\ 0 & Y & 0 \\ 0 & 0 & I \end{bmatrix},$$

and

$$j(\overline{q}_0, u) = \begin{bmatrix} I & 0 & AR \\ 0 & I & S \\ 0 & 0 & I \end{bmatrix} \quad \text{if } u = \begin{bmatrix} I & 0 & R \\ 0 & I & S \\ 0 & 0 & I \end{bmatrix}.$$

We have $\mathscr{H}'_\nu = L^2(\mathscr{M}_k, \mathscr{H}_\sigma \otimes L^2(X))$ and $\mathscr{H}_{\pi'} = L^2(\mathscr{M}_k^* \times S^*, \mathscr{H}_\sigma)$, where

$$X = \left\{ \begin{bmatrix} I & X & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \right\} \tag{56}$$

and

$$S = \left\{ \begin{bmatrix} I & 0 & 0 \\ 0 & I & S \\ 0 & 0 & I \end{bmatrix} \right\}. \tag{57}$$

We identify $\mathscr{M}_k^* \times S^*$ with $\mathscr{M}_k \times X$ via

$$\left\langle (A, \xi), \begin{bmatrix} R \\ \eta \end{bmatrix} \right\rangle = \operatorname{tr}(AR + \xi\eta), \tag{58}$$

where $A \in \mathscr{M}_k$, $\xi \in X$, $R \in \mathscr{M}_k$, $\eta \in S$.

The Hilbert spaces $\mathscr{H}_{\nu'}$ and $\mathscr{H}_{\pi'}$ are canonically isomorphic. Indeed linear combinations of vectors of the form $F(A) = f(A)g \otimes v$ with $f \in L^2(\mathscr{M}_k)$, $g \in L^2(X)$, $v \in \mathscr{H}_\sigma$ are dense in $\mathscr{H}_{\nu'}$.

Analogously linear combinations of vectors of the form $g(A, \xi) = f(A)g(\xi)v$ with $f \in L^2(\mathscr{M}_k^*)$, $g \in L^2(S^*)$ and $v \in \mathscr{H}_\sigma$ are dense in $\mathscr{H}_{\pi'}$.

The canonical isomorphism consists of identifying $F(A)$ and $G(A, \xi)$, using identification (58).

Let m, u be as above. Then we calculate

$$(\nu'(m)F)(A) = |\det Z|^{k/2} |\det W|^{k/2} \chi(Y)(\omega_{\mathrm{tr}}(Z, Y, AX)g) \otimes \sigma(Z)vf(Z^{-1}AW) \tag{59}$$

and

$$(\nu'(u)F)(A) = \omega_{\mathrm{tr}}(AR, S) \otimes vf(A). \tag{60}$$

We then wish to calculate the transpose action of M on $\mathscr{M}_k^* \times S^*$ under our identification. We obtain

$$(A, \xi) \cdot {}^t m = (ZAW^{-1}, -ZAW^{-1}XY^{-1} + Z\xi Y^{-1}). \tag{61}$$

Finally we compute

$$\begin{aligned} \pi'(m)G(A, \xi) = {} & |\det W|^{k/2} |\det Z|^{(n-k)/2} |\det Y|^{k/2} \chi(W)\chi(Y)\chi(Z)^{-1} \\ & \cdot \sigma(Z)vf(Z^{-1}AW)g(Z^{-1}AX + Z^{-1}\xi Y) \end{aligned} \tag{62}$$

and

$$(\pi'(u)G)(A, \xi) = \chi_1(\operatorname{tr}(-AR - \xi S))G(A, \xi). \tag{63}$$

Comparison of (59), (60) with (62), (63) (see also (35), (39)) shows that the two representations are realized by "almost" the same formulae. The only discrepancy is the factor $\chi(W)\chi(Z)^{-1}$ which appears in (63) and not in (60). This can be eliminated easily, however, by use of the unitary operator $T_\chi \colon L^2(\mathscr{M}_k) \to L^2(\mathscr{M}_k)$ given by

$$(T_\chi f)(A) = \chi^{-1}(A)f(A). \tag{64}$$

This concludes the proof of Theorem 4.1. □

The computations in the proof are carried out in detail in [S].

III. ASYMPTOTICS OF MATRIX COEFFICIENTS

1. Notation and basic notions

Let G be a locally compact group with center Z. Fix a compact subgroup K of G. If $\rho \in \operatorname{Rep} G$ and u, v are in the representation space of ρ, denote by $\phi_{u,v}$ the matrix coefficient

$$\phi_{u,v}(g) = (\rho(g)u, v). \tag{1}$$

Definition 1.1. Suppose $\rho \in \operatorname{Rep} G$ with representation space $\mathscr{H}$. Suppose further that Z acts by a character (this is always true if, for example, ρ happens to be irreducible). We say ρ is an $L^p(G/Z)$-representation if there exists a dense subspace $\mathscr{U}$ of $\mathscr{H}$ such that $|\phi_{u,v}| \in L^p(G/Z)$ whenever $u, v \in \mathscr{U}$.

Now let $\rho \in \operatorname{Rep} G$ have representation space $\mathscr{H}$. Write

$$\mathscr{H} = \bigoplus_{\mu \in \hat{K}} \mathscr{H}_\mu, \tag{2}$$

where K acts on $\mathscr{H}_\mu$ by a multiple of μ.

Definition 1.2. Let $\rho \in \operatorname{Rep} G$ and let Φ be a nonnegative function on G satisfying

$$\Phi(g) = \Phi(g^{-1}), \qquad \Phi(zk_1 g k_2) = \Phi(g) \tag{3}$$

whenever $g \in G$, $k_1, k_2 \in K$, $z \in Z$. Let Ψ be a function on $\hat{K}$. We say ρ is (Φ, Ψ)-bounded if whenever $\mu, \nu \in \hat{K}$ and $u \in \mathscr{H}_\mu$, $\nu \in \mathscr{H}_\nu$

$$|\Phi_{u,v}(g)| \leq \Phi(g)\Psi(\mu)\Psi(\nu)\, \|u\|\, \|v\|. \tag{4}$$

Proposition 1.3. *Suppose ρ is (Φ, Ψ)-bounded and $\Phi \in L^p(G/Z)$. Then ρ is an $L^p(G/Z)$-representation.* $\square$

While the definition of (Φ, Ψ)-boundedness is more cumbersome that the definition of L^p-ness of representations, it is easier to work with for our purposes. We give two lemmas which will be useful later.

Lemma 1.4. *Let $\rho = \int_X^\oplus \sigma_x \, d\mu(x)$. Then ρ is (Φ, Ψ)-bounded if and only if σ_x is (Φ, Ψ)-bounded for almost all x.*

Lemma 1.5. *The set of (Φ, Ψ)-bounded representations is closed in* $\operatorname{Rep} G$ *(with respect to the Fell topology).*

The proofs are easy and left to the reader.

We now specialize these notions to the case of general linear groups. We shall write $B_n = \{g \in G_n,\ g_{ij} = 0 \text{ if } i > j\}$, the Borel subgroup of upper triangular matrices, and $A_n = \{g \in G_n,\ g_{ij} = 0 \text{ if } i \neq j\}$, the maximal torus of diagonal matrices.

We shall denote by K_n a maximal compact subgroup of G_n as follows. If $F = \mathbf{R}$, K_n will be the group $\mathrm{O}(n)$ of orthogonal matrices. If $F = \mathbf{C}$, K_n will denote the group $\mathrm{U}(n)$ of unitary matrices. If F is non-Archimedean and $\mathscr{O}$ is its ring of integers, K_n shall denote the group $\mathrm{GL}_n(\mathscr{O})$ of matrices whose entries lie in $\mathscr{O}$ and whose determinant is invertible in $\mathscr{O}$.

We have the Iwasawa decomposition $G_n = B_n K_n$.

Let P be any subgroup of G_n containing B_n. Let $\sigma \in \operatorname{Rep} P$ be realized on $\mathscr{H}_\sigma$. Because of the Iwasawa decomposition, $\operatorname{Ind}_P^{G_n} \sigma$ can be realized on the space $\mathscr{H}$ of functions u from K_n to $\mathscr{H}_0$ satisfying

(1) u is measurable,

(2) If $p \in K_n \cap P$, then $u(pk) = \sigma(p)u(k)$ for all $k \in K_n$.

(3) $\|u(k)\| \in L^2(K_n \cap P \backslash K_n, \mu)$, where μ is the unique K_n-invariant measure on $K_n \cap P \backslash K_n$.

Let $k: G_n \to K_n$ and $b: G_n \to B_n$ be measurable functions such that for all $g \in G_n$

$$g = b(g)k(g). \tag{5}$$

Then a formula for $\operatorname{Ind}_P^{G_n} \sigma$ is

$$((\operatorname{Ind}_P^{G_n} \sigma)(g)u(k') = \lambda(k', g)^{1/2}\sigma(b(k'g))u(k(k'g)), \tag{6}$$

where

$$\lambda(k', g) = \frac{d(\mu \cdot g)}{d\mu}(k'). \tag{7}$$

Note that $\lambda(k', k) = 1$ for all $k \in K_n$.

It is easy to check that $u \in \mathscr{H}$ is K_n-invariant if and only if $u(k) = \overline{u}$ for all $k \in K_n$ with $\overline{u}$ a $K_n \cap P$-invariant vector in $\mathscr{H}_\sigma$. Let U_n be the unipotent radical of B_n, i.e. the group

$$U_n = \{g \in B_n,\ g_{ii} = 1 \text{ for all } i\}.$$

The torus A_n acts on U_n by conjugation. Let Σ^+ be the set of positive roots with respect to this action:

$$\Sigma^+ = \{\alpha: A \to F^\times,\ \exists u \in U_n - \{I\} \text{ s.t. } u^a = \alpha(a)(u - I) + I\}. \tag{8}$$

The positive Weyl chamber in A_n, denoted by A_n^+, is

$$A_n^+ = \{a \in A_n,\ |\alpha(a)| \geq 1 \text{ for all } \alpha \in \Sigma^+\}. \tag{9}$$

It is very easy to make the above explicit:

$$\Sigma^+ = \{\alpha_{ij},\ i > j\}, \tag{10}$$

where

$$\alpha_{ij}(\operatorname{diag}(a_1, \dots, a_n)) = a_i a_j^{-1} \tag{11}$$

so that

$$A_n^+ = \{\operatorname{diag}(a_1, \dots, a_n),\ |a_1| \leq \cdots \leq |a_n|\}. \tag{12}$$

It is easy to check that the Radon-Nikodým derivative of the action of G_n on $K_n \cap B_n \backslash K_n$ is given by

$$\lambda(k', g) = \delta_{B_n}(b(g)), \tag{13}$$

where δ_{B_n}, the modular function of B_n, is the group morphism $B_n \to \mathbf{R}^+$ determined by

$$\begin{aligned} \delta_{B_n}(a) &= \prod_{\alpha \in \Sigma^+} |\alpha(a)| \quad \text{for } a \in A_n, \\ \delta_{B_n}(u) &= 1 \quad \text{for } u \in U_n. \end{aligned} \tag{14}$$

Let τ denote the trivial representation of B_n. Up to multiples, there is a unique K_n-invariant vector u in $\operatorname{Ind}_{B_n}^{G_n} \tau$, given by $u(k) = 1$. Its matrix coefficient Ξ_n, called the Harish-Chandra spherical function, is given by

$$\Xi_n(g) = \int_{K_n} \delta^{1/2}(b(kg))\, dk. \tag{15}$$

Proposition 1.6 [HC, Si]. *There exist constant c_1, $c_2(\varepsilon)$ such that for all $\alpha \in A_n^+$ and $\varepsilon > 0$*

$$c_1 \delta_{B_n}^{-1/2}(a) \le \Xi_n(a) \le c_2(\varepsilon) \delta_{B_n}^{-1/2+\varepsilon}(a). \quad \square \tag{16}$$

We have the Cartan decomposition $G_n = K_n A_n^+ K_n$.

Proposition 1.7. *Haar measure on G_n with respect to the Cartan decomposition is given by*

$$dg = \Delta(a) dk_1 da\, dk_2, \tag{17}$$

where $\Delta(a)$ is a positive function on A_n^+ satisfying

$$d_1(t) \delta_{B_n}(a) \le \Delta(a) \le d_2 \delta_{B_n}(a) \tag{18}$$

for all $a \in \{\alpha \in A_n^+,\ |\alpha(a)| \ge t$ for all $\alpha \in \Sigma^+\}$ and for some constants d_2 and $d_1(t)$ with $d_2(t) > 0$ if $t > 1$.

For the proof, see e.g. [W].

Corollary 1.8. *If $\rho \in \operatorname{Rep} G_n$ is (Ξ_n^r, Ψ)-bounded for some $r > 0$ and some Ψ, then ρ is a $L^{2/r+\varepsilon}(G_n/Z)$-representation for all $\varepsilon > 0$.*

Finally, we state an important result we shall need in the sequel.

Theorem 1.9. *The regular representation λ of G_n (and hence, by Lemma 1.5 any representation in the closure in $\operatorname{Rep} G_n$ of $\{\lambda\}$) is $(\Xi_n, \dim \mu)$-bounded.*

For the proof, see [H1].

2. Restricting representations

Definition 2.1. Let $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_n$. The standard embedding ϕ_I of G_k in G_n corresponding to I is the injection such that the map from F^k to F^n given by $e_j \to e_{i_j}$ for $j = 1, \ldots, k$ intertwines the standard action of G_k on F^k and the action of G_k on F^n induced by ϕ_I.

Theorem 2.2. *Suppose $\rho \in \operatorname{Rep} G_n$ has pure rank $k \neq n/2$. Then for any standard embedding of G_{k+1}, $\rho|_{G_{k+1}}$ is $(\Xi_{k+1}, \dim\mu)$-bounded. If $k = n/2$ (in particular n is even), $\rho|_{G_k}$ is $(\Xi_k, \dim\mu)$-bounded for any standard embedding of G_k.*

Note. It is easy to see from the definition of δ_{B_n} in Proposition 1.6 that $\Xi_k \geq \Xi_{k+1}|_{G_k}$. Hence the result for $k = n/2$ is weaker than the others.

This does not arise from a defect in the proof, but from an actual feature of representation theory of G_n, namely the existence of the complementary series.

Let χ_α be the nonunitary character of $P_k^{(2k)}$ which is trivial on $N_k^{(2k)}$ and given by

(20) $$\chi_\alpha = |\det|^\alpha \otimes |\det|^{-\alpha}, \qquad \alpha \in \mathbf{R},$$

on the Levi component of $P_k^{(2k)}$ (which is isomorphic to $G_k \times G_k$).

It is well known [St] that $\operatorname{Ind}_{P_k}^{G_{2k}} \chi_\alpha$ can be "unitarized" if $-\frac{1}{2} < \alpha < \frac{1}{2}$, giving rise to a family of unitary representations of G_{2k}, called the complementary series.

These representations are easily seen to have rank k (for example this follows from the results of Chapter II).

It is also easy to see that $\operatorname{Ind}_{P_k}^{G_{2k}} \chi_\alpha|_{G_{k+1}}$ are not (Ξ_{k+1}, Ψ)-bounded for any Ψ, thus giving a counterexample to the possible cleaner version of Theorem 2.2.

The rest of this section is dedicated to the proof of Theorem 2.2

First of all, since all the images of standard embeddings of G_{k+1} (respectively G_k) are conjugate, it is clearly enough to consider the case $I = \{1, \ldots, k+1\}$ (respectively $I = \{1, \ldots, k\}$).

First suppose $k \neq n/2$. Consider the subgroup $P_{k+1}^{(n)}$. We want to analyze representations of $P_{k+1}^{(n)}$ of rank k from the point of view of their restriction of G_{k+1}.

We identified $\widehat{N}_{k+1}$ with $\operatorname{Hom}_F(F^{k+1}, F^{n-(k+1)})$. Let

(21) $$\mathscr{V} = \{\text{subspaces of } F^{n-(k+1)} \text{ of dimension } k\}.$$

G_{k+1} acts on $\operatorname{Hom}_F(F^{k+1}, F^{n-(k+1)})$ by right multiplication. Hence the $G_{k+1} \times G_{n-(k+1)}$-orbit $\mathscr{O}_k$ in $\widehat{N}_{k+1}$ decomposes into the G_{k+1}-orbits

(22) $$\mathscr{O}_V = \{T \in \operatorname{Hom}_F(F^{k+1}, F^{n-(k+1)}), \operatorname{Im} T = V\}$$

as V varies in $\mathscr{V}$. Hence if $\rho \in \operatorname{Rep} P_{k+1}$ has rank k,

(23) $$\rho|_{G_{k+1}} = \int_{\mathscr{V}}^{\oplus} \rho_V \, d\mu(V).$$

The ρ_V are all isomorphic. Indeed $G_{n-(k+1)}$ acts transitively on $\mathscr{V}$ and commutes with G_{k+1}.

Since (Φ, Ψ)-boundedness is preserved by direct integrals (Lemma 1.4) it will be enough to prove $(\Xi_{k+1}, \dim \mu)$-boundedness for a single ρ_V. To fix ideas, let

$$V = \operatorname{span}\{e_{k+2}, \dots, e_{2k+1}\}. \tag{24}$$

Let $T: e_i \mapsto e_{i+k+1}$ for $1 \le i \le k$ and $T(e_{k+1}) = 0$. Then $T \in \mathscr{O}_V$. Let

$$S = \operatorname{stab}_{G_{k+1}}(T) = \{g \in G_{k+1},\ g_{ij} = \delta_{ij} \text{ unless } j = k+1\}. \tag{25}$$

Theorem 1.1 of [R] says that any unitary representation of $G_{k+1}N_{k+1}$ whose spectral measure is concentrated on $\mathscr{O}_V$ is induced from a representation of SN_{k+1} whose restriction to N_{k+1} is a multiple of the character χ_T corresponding to T.

Since SN_{k+1} is a semidirect product, χ_T extends to SN_{k+1} via the trivial representation of S. Again [R] says that

$$\rho_V = \sigma \otimes \chi_T \tag{26}$$

for some $\sigma \in \operatorname{Rep} S$. Using the Mackey decomposition theorem [M2] it is easy to check that since $G_{k+1} \backslash G_{k+1}N_{k+1}/SN_{k+1}$ has a single element,

$$(\operatorname{Ind}_{SN_{k+1}}^{G_{k+1}N_{k+1}} \sigma \otimes \chi_T)|_{G_{k+1}} = \operatorname{Ind}_S^{G_{k+1}} \sigma. \tag{27}$$

In conclusion, we must show that $\sigma \in \operatorname{Rep} S$, $\operatorname{Ind}_S^{G_{k+1}} \sigma$ is $(\Xi_{k+1}, \dim \mu)$-bounded. This can be shown directly. However, it is easier to check this is true by using a few general facts.

Since $S \simeq F^\times \times F^k$ is solvable, σ is weakly contained in the regular representation of S [Wg]. Since induction is a continuous functor [F2], $\rho_V = \operatorname{Ind}_S^{G_{k+1}} \sigma$ is weakly contained in the regular representation of G_{k+1}. Theorem 1.9 now implies the required estimate.

The case where $k = n/2$ is considerably easier. The group G_k now acts simply transitively on the orbit $\mathscr{O}_k \subseteq \widehat{N}_k$. Again [R] shows that

$$\rho|_{G_k N_k} = \operatorname{Ind}_{N_k}^{G_n N_k} n\chi_v \tag{28}$$

for any $\chi_v \in \mathscr{O}_k$ and some $n \in \mathbf{N} \cup \{\infty\}$. Again, since $G_k \backslash G_k N_k / N_k$ has a single element,

$$\rho|_{G_k} = \operatorname{Ind}_{\{1\}}^{G_k} n \cdot (\text{trivial}) \tag{29}$$

which is a multiple of the regular representation of G_k. Again Theorem 1.9 implies the desired estimate. $\square$

3. Estimates

We wish to use Theorem 2.2 to prove estimates on the matrix coefficients of representations of a fixed rank. Our main result will be:

Theorem 3.1. *Suppose $\rho \in \widehat{G}_n$ satisfies: $\rho|_{G_k}$ is (Ξ_k, Ψ)-bounded for some function Ψ on $\widehat{K}_k$. Write $n = qk + r$ with $q \in N$ and $0 \le r < k$. If x is an integer define*

$$\phi(x) = \begin{cases} x^2 & \text{if } x \text{ is even}, \\ x^2 - 1 & \text{if } x \text{ is odd}. \end{cases}$$

Let

$$e(n, k) = \frac{k-1}{qk-1} \frac{\phi(qk) + \phi(r)}{\phi(n)}.$$

Then for all $\varepsilon > 0$, ρ is $(\Xi_n^{e(n,k)-\varepsilon}, \Psi'_\varepsilon)$-bounded for some function Ψ'_ε on $\widehat{K}_n$.

Corollary 3.2. *Under the hypothesis and notation for Theorem 3.1, ρ is an $L^{2/(e(n,k)+\varepsilon}(G_n/Z)$-representation.*

Proof. Combine Theorem 3.1 and Corollary 1.8. □

When we combine Theorem 3.1, Corollary 3.2 and Theorem 2.2 we obtain:

Theorem 3.3. *Suppose $\rho \in \operatorname{Rep} G_n$ has pure N-rank k. If $k \ne n/2$, then ρ is $(\psi_n^{e(n,k+1)-\varepsilon}, \Psi'_\varepsilon)$-bounded and is an $L^{2/e(n,k+1)+\varepsilon}(G_n/Z)$-representation.*

If $k = n/2$, then ρ is $(\Xi_n^{e(n,k)-\varepsilon}, \Psi'_\varepsilon)$-bounded and is an $L^{2/e(n,k)+\varepsilon}(G_n/Z)$-representation. □

When $r = 0$ (or $n = 2k - 1$), Theorem 3.1 gives a much cleaner form for $e(n, k)$, namely $e(n, k) = (k-1)/(n-1)$. There are strong reasons to believe this estimate to be valid for all n, k and not just when k divides n. For one, the full power of the hypothesis is not quite used in the proof of Theorem 3.1 when $r \ne 0$. Furthermore, explicit calculations (e.g. for $GL_5(\mathbf{R})$) show that the sharper estimate should be valid. Our theorem is not too far off, though. It is easy to see that in the worst case $(r = k-1)$, $e(n, k)$ is about 20% smaller than $(k-1)/(n-1)$.

We formally state

Conjecture 3.4. Theorem 3.1, hence Corollary 3.2 and Theorem 3.3, are valid with $e(n, k)$ replaced by $(k-1)/(n-1)$.

We now prove Theorem 3.1. Proposition 1.6 related the asymptotics of Ξ_n and the modular function δ_{B_n}. During the proof we shall actually compare the various δ_B's instead of the Ξ's. Let α_j be the jth simple root:

$$\alpha_j(\operatorname{diag}(a_1, \dots, a_n)) = a_{j+1} a_j^{-1}, \qquad j = 1, \dots, n-1. \tag{30}$$

Then it is easy to compute

$$\delta_{B_n} = \prod_{l=1}^{n-1} |\alpha_l|^{l(n-l)}. \tag{31}$$

The proof shall proceed in steps.

(a) First suppose $n = 2k$. If $j = 0$ or 1, let

$$I_j = \{i,\ 0 < i \le n,\ i \equiv j \pmod 2\} \tag{32}$$

and $G_k^{(j)} = \Phi_{I_j}(G_k)$ (Definition 2.1).

We shall use superscripts $^{(j)}$ for the images under Φ_{I_j} of all the objects defined in relations to G_k (e.g. $\Xi_k^{(j)}$, $K_k^{(j)}$, etc.).

By hypothesis, $\rho|_{G_k^{(j)}}$ is $(\Xi_k^{(j)}, \Psi)$-bounded for $j = 0, 1$ and some Ψ. Proposition 6.3 of [H1] now implies that $\rho|_{G_k^{(0)} \times G_k^{(1)}}$ is $(\Xi_k^{(0)}\Xi_k^{(1)}, \tilde{\psi})$-bounded for some function $\tilde{\psi}$ on $(K_k^{(0)} \times K_k^{(1)})^\wedge$. (It is well known, see e.g. [H4], that G_n and K_n satisfy the hypotheses of this proposition.) $A_k^{+(0)} \times A_k^{+(1)}$ contains A_n^+. We define a function Φ on G_n by letting it coincide with $\Xi_k^{(0)}\Xi_k^{(1)}$ on A_n^+ and making it invariant by right and left translations by K_n.

I claim ρ is (Φ, Ψ')-bounded for some function Ψ' on $\widehat{K}_n$. Let μ be a K_n-type. As a representation of $K_k^{(0)} \times K_k^{(1)}$, $\mu = \bigoplus m_i\mu_i$, a finite direct sum of irreducible representations of $K_k^{(0)} \times K_k^{(1)}$. Define

$$\Psi'(\mu) = \max \widetilde{\Psi}(\mu_i). \tag{33}$$

If $a \in A_n^+$, u is of type μ, and v is of type ν, then write $u = \sum u_i$, $v = \sum v_j$ corresponding to the decompositions $\mu = \bigoplus m_i\mu_i$, $\nu = \bigoplus n_j\nu_j$ of μ, ν into $K_k^{(0)} \times K_k^{(1)}$-types. Then

$$\begin{aligned} |(\rho(a)u, v)| &\le \sum_{i,j} |(\rho(a)u_i, v_j)| \le \Phi(a) \sum_{i,j} \|u_i\|\,\|v_i\| \widetilde{\Psi}(\mu_i)\widetilde{\Psi}(\nu_j) \\ &\le \Phi(a)\Psi'(\mu)\Psi'(\nu)\|u\|\,\|v\|. \end{aligned} \tag{34}$$

It remains to compare Φ and Ξ_n. Proposition 1.6 states that on A_n^+,

$$c_1\delta_{B_n}^{-1/2} \le \Xi_n \le c_2(\varepsilon)\delta_{B_n}^{-1/2+\varepsilon} \tag{35}$$

and

$$c_3\delta_{B_k^{(0)}}^{-1/2}\delta_{B_k^{(1)}}^{-1/2} \le \Phi \le c_4(\varepsilon)\delta_{B_k^{(0)}}^{-1/2+\varepsilon}\delta_{B_k^{(1)}}^{-1/2+\varepsilon} \tag{36}$$

for appropriate constants c_1, $c_2(\varepsilon)$, c_3, $c_4(\varepsilon)$. Hence to finish the proof of step (a) it suffices to show

$$\delta_s = \delta_{B_k^{(0)}}\delta_{B_k^{(1)}} \ge \delta_{B_n}^{(k-1)/(2k-1)}. \tag{37}$$

Write δ_s as a product of simple roots:

$$\delta_s = \prod_{j=1}^{n-1} |\delta_j|^{s_j}. \tag{38}$$

It is easy to compute

$$s_j = \begin{cases} 2p(k-p) & \text{if } j = 2p\,, \\ p(k-p) + (p+1)(k-p-1) & \text{if } j = 2p+1\,. \end{cases} \tag{39}$$

The maximum of $(n-j)j/s_j$ for $j = 1, \dots, n-1$ occurs for $j = 0$, $2k-2$ and is $(2k-1)/(k-1)$. We are comparing δ_s and δ_{B_n} in the region $\{a, |\alpha_j(a)| \geq 1\}$, so this is enough to prove (37). This concludes the proof of step (a).

(b) We now apply the same method to the case $n = qk$, q arbitrary. If $j = 0, \dots, q-1$, set

$$I_j = \{i,\ 0 < i \leq n,\ i \equiv j \pmod q\}. \tag{40}$$

We use superscripts ${}^{(j)}$ in the same way as in step (a). The same argument as in that step shows that we have to compare

$$\delta_s = \prod_{j=0}^{q-1} \delta_{B_k^{(j)}} \tag{41}$$

with δ_{B_n}. We must show

$$\delta_s \geq \delta_{B_n}^{(k-1)/(qk-1)}. \tag{42}$$

If $0 \leq i < j \leq q-1$, let

$$\delta_{ij} = \delta_{B_k^{(j)}} \delta_{B_k^{(j)}} \tag{43}$$

and

$$\delta_{i\cup j} = \delta_{B_{2k}^{(i\cup j)}}, \tag{44}$$

where

$$G_{2k}^{(i\cup j)} = \phi_{I_i \cup I_j}(G_{2k}). \tag{45}$$

By step (a)

$$\delta_{ij} \leq \delta_{i\cup j}^{(k-1)/(2k-1)}. \tag{46}$$

As we take the product of (46) over al pairs i, j the left-hand side yields δ_s^{q-1}. As for the right-hand side, it is easy to see that as we take the product each positive root appears only once, except for the roots in $G_k^{(i)}$ for some i. These appear $q-1$ times. Hence

$$\delta_s^{q-1} \geq (\delta_{B_n} \cdot \delta_s^{q-2})^{(k-1)/(2k-1)}, \tag{47}$$

which is equivalent to (42).

(c) Now suppose $n = k+r$ and $\rho|_{G_k}$ is (Ξ_k, Ψ)-bounded. If G_r is embedded in G_k in any standard way, it is clear from 1.6 and (31) that

$$\Xi_k|_{A_r^+} < \delta_{B_r}^{-\frac{1}{2}(k-1)/(r-1)+\varepsilon}. \tag{48}$$

Hence for u, v in the dense subspace of K_k-finite vectors, $\phi_{u,v} \in L^2(G_r/Z)$. Hence $\rho|_{G_r}$ can be embedded in $\infty\mathrm{reg}_{G_r/Z}$. Since (Φ, Ψ)-boundedness is clearly preserved under taking direct sums and subrepresentations, Theorem 1.9 shows that $\rho|_{G_r}$ is $(\Xi_r, \dim\mu)$-bounded.

Let

$$I = \{[(k-r)/2] + 2u,\ u = 1, \ldots, r\} \tag{49}$$

and let J be the complement of I in $\{1, \ldots, k+r\}$. Define

$$\widetilde{G}_r = \phi_I(G_r) \tag{50}$$

and

$$\widetilde{G}_k = \phi_J(G_k). \tag{51}$$

The same argument as in step (a) shows that we have to prove

$$\delta_s = \delta_{\widetilde{B}_k}\delta_{\widetilde{B}_r} \geq \delta_{B_n}^{(\phi(k)+\phi(r))/\phi(n)}, \tag{52}$$

and again the estimate will be proved if we can show that if

$$\delta_s = \prod_{i=1}^{n-1} |\alpha_i|^{s_i}, \tag{53}$$

then

$$\frac{(n-i)i}{s_i} \leq \frac{\phi(n)}{\phi(k)+\phi(r)}. \tag{54}$$

One can compute that if n is even, then

$$s_i = i(k-i) \quad \text{if } i \leq (k-r)/2, \tag{55}$$

(56)
$$s_i = \begin{cases} \frac{1}{4}(k^2 - (r-2u)^2) + (r-u)u & \text{if } i = (k-r)/2 + 2u, \\ \frac{1}{4}(k^2 - (r-2u)^2) + (r-u-1)(u+1) & \text{if } i = (k-r)/2 + 2u + 1 \end{cases}$$

when $(k-r)/2 \leq i \leq (k+3r)/2$ and

$$s_i = (k+r-i)(i-r) \quad \text{if } i \geq (k+3r)/2. \tag{57}$$

On the other hand, if n is odd,

$$s_i = i(k-i) \quad \text{if } i \leq (k-r+1)/2 \tag{58}$$

$$s_i = \begin{cases} \frac{1}{4}(k^2 - (r-2u-1)^2) + (r-u)u & \text{if } i = (k-r+1)/2 + 2u, \\ \frac{1}{4}(k^2 - (r-2u-1)^2) + (r-u-1)(u+1) & \\ & \text{if } i = (k-r+1)/2 + 2u + 1 \end{cases} \tag{59}$$

when $(k-r+1)/2 \leq i \leq (k+3r+1)/2$ and

$$s_i = (k+r-i)(i-r) \quad \text{if } i \geq (k+3r-1)/2. \tag{60}$$

Elementary techniques now show that the maximum of $(n-i)i/s_i$ occurs at $i=[n/2]$ and is $\phi(n)/(\phi(k)+\phi(r))$. This proves step (c).

(d) Finally let us prove the general case. For any standard embedding of G_{qk} in G_n, $\rho|_{G_{qk}}$ will be $(\Xi^{(qk-1)/(k-1)-\varepsilon}, \Psi)$-bounded for some Ψ by step (b). The same argument as in step (c) shows that for any standard embedding of G_r into G_n, $\rho|_{G_r}$ is $(\Xi_r, \dim\mu)$-bounded, hence certainly $(\Xi_r^{(qk-1)/(k-1)-\varepsilon}, \dim\mu)$-bounded. Now exactly the same computations as in step (c) give the desired result, since in the proof of step (c) we just compared Ξ-functions. This concludes the proof of Theorem 3.1. □

Our last result shows how rank may be used to derive information about the topology of $\widehat{G}_n$.

Theorem 3.5. *Let $k \le (n-2)/3$ and $n > 2$. Then the set $(\widehat{G}_n)_k$ of representations of N-rank k is both open and closed in $\widehat{G}_n$. In particular, for $n>2$, G_n has (a property closely related to) Kazhdan's property* (T). *Namely, the connected component of the trivial representation in $\widehat{G}_n$ consists of characters.*

Proof. Since restriction defines a continuous map from $\widehat{G}_n$ to $\operatorname{Rep} N_{[n/2]}$, it is obvious that $\bigcup_{l\le k}(\widehat{G}_n)l$ is closed, so that it is enough to prove that under our assumptions $(\widehat{G}_n)_k$ is open. This is equivalent to showing that if $\rho \in (\widehat{G}_n)_k$ and σ is a direct integral of representations in $(\widehat{G}_n)_l$, $l>k$, then ρ is not weakly contained in σ.

Define a representation ω_m of G_m as follows: the representation space of ω_m is $L^2(F^m)$. If $g \in G_m$ and $f \in L^2(F^m)$, we define

$$(\omega_m(g)f)(x) = |\det g|^{1/2} f(xg). \tag{61}$$

Then it is obvious that under any standard embedding of G_l in G_m,

$$\omega_m|_{G_l} = \infty\omega_l. \tag{62}$$

The proof of Theorem I.3.6 shows that if $\tau \in \widehat{G}_m$, $\operatorname{rk}\tau < [m/2]$, then $\tau|_{G_{m-2}}$ is a direct integral of representations of the form $\nu \otimes \omega_{m-2}$, with $\operatorname{rk}\nu = \operatorname{rk}\tau - 1$.

An inductive argument that uses this fact and (62) shows that under our assumptions on ρ,

$$\rho|_{G_{n-2k}} = \infty\omega_{n-2k}^{\otimes k}. \tag{63}$$

Since $n-2k \ge k+2$, again by (62),

$$\rho|_{G_{k+2}} = \infty\omega_{k+2}^{\otimes k}. \tag{64}$$

A straightforward calculation shows that $\omega_m^{\otimes l}$ is (Ξ_m, Φ)-bounded exactly if $l \ge m-1$. Hence $\rho|_{G_{k+2}}$ is not (Ξ_{k+2}, Φ)-bounded.

On the other hand, Theorem 2.2 and Lemma 1.4 show that $\sigma|_{G_{k+2}}$ is $(\Xi_{k+2}, \dim \mu)$-bounded.

The result now follows easily from the fact that restriction to G_{k+2} defines a continuous map $\widehat{G}_n \to \operatorname{Rep} G_{k+2}$ and from Lemma 1.5. □

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